

VON NEUMANN SPECTRA NEAR THE SPECTRAL GAP

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Abstract. In this paper we study some new von Neumann spectral invariants associated to the Laplacian acting on L^2 differential forms on the universal cover of a closed manifold. These invariants coincide with the Novikov-Shubin invariants whenever there is no spectral gap in the spectrum of the Laplacian, and are homotopy invariants in this case. In the presence of a spectral gap, they differ in character and value from the Novikov-Shubin invariants. Under a positivity assumption on these invariants, we prove that certain L^2 theta and L^2 zeta functions defined by metric dependent combinatorial Laplacians acting on L^2 cochains associated with a triangulation of the manifold, converge uniformly to their analytic counterparts, as the mesh of the triangulation goes to zero.

Introduction

Let (M, g) be a compact connected Riemannian manifold and \widetilde{M} be its universal cover, which is a

Riemannian manifold with the induced metric. Although we only consider the universal cover in this paper, our results extend easily to any Galois covering of M . Let $\Delta_j = d\delta + \delta d$ denote the Laplacian acting on L^2 differential j -forms on \widetilde{M} , where d denotes the de Rham differential and δ its adjoint. Let $\lambda_{0,j}$ denote the bottom of the spectrum of Δ_j on the orthogonal complement of its kernel.

Let $k_j(t, x, y)$ denote the integral kernel of the heat operator $e^{-t\Delta_j}$. By parabolic regularity theory, $k_j(t, x, y)$ is smooth. Since Δ_j commutes with the action of the fundamental group $\pi_1(M)$, we see that

$$k_j(t, \gamma.x, \gamma.y) = k_j(t, x, y)$$

for all $\gamma \in \pi_1(M)$ (here we identify the cotangent spaces of x and $\gamma.x$ via γ). Recall that the von Neumann trace of $e^{-t\Delta_j}$ is given by

$$\tau(e^{-t\Delta_j}) = \int_M \text{tr}(k_j(t, x, x)) dx.$$

Atiyah [2] defined the L^2 Betti numbers of M as follows,

$$b_{(2)}^j(\widetilde{M}) = \tau(P_j),$$

where P_j denotes the orthogonal projection onto the kernel of Δ_j . Then, since $k_j(t, x, y)$ converges as $t \rightarrow \infty$ uniformly over compact subsets to $k_{P_j}(x, y)$, where $k_{P_j}(x, y)$ denotes the integral kernel of the operator P_j (cf. [20]), we see that

$$b_{(2)}^j(\widetilde{M}) = \lim_{t \rightarrow \infty} \tau(e^{-t\Delta_j}).$$

Define the von Neumann algebra theta functions

$$\theta_j(t) = \tau(e^{-t\Delta_j}) - b_{(2)}^j(\widetilde{M})$$

Explicit calculations tend to show that the large time asymptotics of $\theta_j(t)$ are of the form $e^{-\lambda_{0,j}t}t^{-\beta}$ for some $\beta > 0$. This motivates the following definitions

$$\beta_j(M, g) = \sup\{\beta \in \mathbb{R} : e^{\lambda_{0,j}t}\theta_j(t) \text{ is } O(t^{-\beta}) \text{ as } t \rightarrow \infty\} \in [0, \infty]$$

$$\overline{\beta}_j(M, g) = \inf\{\beta \in \mathbb{R} : t^{-\beta} \text{ is } O(e^{\lambda_{0,j}t}\theta_j(t)) \text{ as } t \rightarrow \infty\} \in [0, \infty].$$

Whenever $\lambda_{0,j} = 0$ (note that this condition is independent of the choice of metric cf. [12]), $\beta_j(M, g)$ and $\overline{\beta}_j(M, g)$ coincide with the Novikov-Shubin invariants (cf. [10], [12], [3], [14]) of M and therefore they are independent of the choice of metric. However when $\lambda_{0,j} > 0$ general, they differ from the Novikov-Shubin invariants, and are probably dependent on the choice of Riemannian metric, as an example by Donald Cartwright [5] appears to indicate. But $\beta_j(M, g)$ and $\overline{\beta}_j(M, g)$ are by fiat clearly von Neumann spectral invariants, which tend to be finite even when Novikov-Shubin invariants aren't, as we demonstrate by examples in section 5. They are also invariant under scaling of the metric by a constant. The Hodge star operator on the universal cover of M is a von Neumann algebra isometry intertwining the Laplacians Δ_j and Δ_{n-j} , where n denotes the dimension of M . Hence we see that $\beta_j(M, g) = \beta_{n-j}(M, g)$ and $\overline{\beta}_j(M, g) = \overline{\beta}_{n-j}(M, g)$ for all $0 \leq j \leq n$. Using the local Harnack inequality [24], one observes that the

large time asymptotics of $\theta_0(t)$ coincide with the large time asymptotics of $\|e^{-t\Delta_0}\|_{1 \rightarrow \infty}$. It is unclear to us if this is also true on j -forms, where $j > 0$.

We next give a summary of our results. Let K denotes a triangulation of M . In section 1, we define a metric dependent combinatorial Laplacian $\Delta_j^{\tilde{K}, \tilde{W}}$ acting on L^2 cochains on the universal cover \tilde{K} , besides other background material. Section 2 contains a detailed analysis of the relationship between the spectral density functions for the combinatorial Laplacian and the analytic Laplacian. These estimates are the analogues on non-compact covering spaces of some results of Dodziuk and Patodi [8] that also enable us to show that the bottom of the spectrum of the combinatorial Laplacian, $\lambda_{0,j}^{\tilde{K}, \tilde{W}}$ converges to the bottom of the spectrum of the analytic Laplacian $\lambda_{0,j}$, as the mesh of the triangulation goes to zero. In section 3 we prove some estimates for the combinatorial and the analytic L^2 theta functions, which are used later on in the paper. In section 4, we prove that $e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t)$ converges uniformly in $t \in [t_0, \infty)$ to $e^{\lambda_{0,j} t} \theta_j(t)$, for any $t_0 > 0$, as the mesh η of the triangulation K goes to zero. In section 5, we compute the values of the invariants $\beta_j(M, g), \tilde{\beta}_j(M, g)$ for closed odd dimensional hyperbolic manifolds. In section 6, under the assumption that $\beta_j(M, g) > 0$, we define the von Neumann determinant of the operator $\Delta_j - \lambda_{0,j}$ as well as the analytic β torsion, which is defined analogously to Ray-Singer torsion [18], as an alternating product of the von Neumann determinants of $\Delta_j - \lambda_{0,j}$. We also define its combinatorial counterpart, which we call combinatorial β -torsion. In section 7, we prove convergence theorems for L^2 theta and L^2 zeta functions respectively, as the mesh of the triangulation goes to zero, extending to the non-compact covering space classical theorems [8]. The latter arise naturally in the study of L^2 torsion invariants, see [16, 4] and [14]. We anticipate that the technical results of this paper have applications in a number of directions and will provide the groundwork for the investigation of other spectral properties of the Laplacian on non-compact manifolds. The results of section 7 also give evidence that the combinatorial β -torsion converges to the analytic β -torsion, as the mesh of the triangulation goes to zero. We plan to study this and other questions raised in this paper elsewhere.

We conclude the introduction with the conjecture that $\beta_j(M, g)$ is always positive. Earlier, Sunada [23] had conjectured that this was the case for functions, that is, $\beta_0(M, g) > 0$. This has been proved by Terry Lyons and we present his proof in the appendix to our paper. We also conjecture that $\beta_j(K, g)$ converges to $\beta_j(M, g)$ as the mesh of the triangulation goes to zero, where $\beta_j(K, g)$ denotes the combinatorial counterpart of $\beta_j(M, g)$. The results of section 4 suggest that this conjecture is true, and one can find a discussion of it over there.

1 Preliminaries.

In this section we establish the notation of the paper. We recall the definition of the combinatorial and analytic L^2 theta functions and also some of their basic properties following the notation established in [4] and [7] (with some minor changes). Thus (M, g) is a compact manifold without boundary with Riemannian metric g and dimension n . The fundamental group of M we denote by Γ and we lift g to a Γ invariant Riemannian metric \tilde{g} on the universal cover \tilde{M} . Now introduce the space $\Omega_{(2)}(\tilde{M})$ of L^2

differential forms with respect to the volume defined by this metric. Denote by d the exterior derivative (which is a Γ -invariant lift of the exterior derivative on forms on M) on $\Omega_{(2)}(\widetilde{M})$. Strictly speaking we must restrict the action of d to sufficiently smooth forms. When this becomes crucial we shall introduce the appropriate notation. Let K be a triangulation of M and \widetilde{K} be the induced Γ -invariant triangulation of \widetilde{M} . Let $C(K)$ denote the space of cochains on K and $C_{(2)}(\widetilde{K})$ denote the space of ℓ^2 cochains on \widetilde{K} .

Following [7], we use the de Rham and Whitney maps to relate the combinatorial and de Rham complexes. For \widetilde{M} , let \widetilde{A} denote the de Rham map from the de Rham complex to the combinatorial complex and \widetilde{W} denote the Whitney map which goes in the reverse direction and satisfies:

$$\widetilde{A}\widetilde{W} = 1$$

(we will review the definitions of these maps in section 2). We may use the Whitney map to define an equivalent inner product on $C_{(2)}(\widetilde{K})$ by setting

$$\langle c_\sigma, c_{\sigma'} \rangle_{\widetilde{W}} = \langle \widetilde{W}c_\sigma, \widetilde{W}c_{\sigma'} \rangle$$

where σ, σ' are elements of \widetilde{K} and c_σ denotes the characteristic cochain of σ . The fundamental group Γ acts on each of the complexes $\Omega_{(2)}(\widetilde{M})$ and $C_{(2)}(\widetilde{K})$ by unitary operators. There are isomorphisms of Γ -modules:

$$\Omega_{(2)}(\widetilde{M}) \cong \Omega_{(2)}(M) \otimes \ell^2(\Gamma), \quad C_{(2)}(\widetilde{K}) \cong C(K) \otimes \ell^2(\Gamma),$$

where $\Omega_{(2)}(M)$ denotes L^2 forms on M and Γ acts on $\ell^2(\Gamma)$ by the left regular representation λ . The von Neumann algebra generated by $\{\lambda(\gamma) \mid \gamma \in \Gamma\}$ is a finite von Neumann algebra \mathcal{U} with normalised trace denoted τ . We may regard the trace τ as being defined on the commutant as well (as in [6]) by letting χ_e denote the function in $\ell^2(\Gamma)$ which is one at the identity of Γ and zero elsewhere and then $\tau(B) = \langle B\chi_e, \chi_e \rangle_{\ell^2(\Gamma)}$, for any B in \mathcal{U} or \mathcal{U}' . There is also a finite trace on the commutant $\mathcal{B}(C(K)) \otimes \mathcal{U}'$ of this \mathcal{U} action on $C_{(2)}(\widetilde{K})$: it is $\text{tr} \otimes \tau$ where tr denotes the usual matrix trace on the finite dimensional space $\mathcal{B}(C(K))$. Note that \mathcal{U} and \mathcal{U}' are anti-isomorphic [6].

There is a semifinite trace also denoted $\text{tr} \otimes \tau$ on the commutant of the \mathcal{U} action on $\Omega_{(2)}(\widetilde{M})$. This commutant is just $\mathcal{B}(\Omega_{(2)}(M)) \otimes \mathcal{U}'$ so that ‘tr’ represents the unique semifinite trace on $\mathcal{B}(\Omega_{(2)}(M))$ which extends the usual trace on the trace class operators. Where no confusion can arise we simply write τ for any of these traces.

Now we introduce the spectral density functions for these complexes. Let d_j denote the restriction of d to j -forms, with a similar convention for d_j^K and $d_j^{\widetilde{K}}$ acting on $C^j(K)$ and $C_{(2)}^j(\widetilde{K})$ respectively. Let δ_j denote the Hilbert space adjoint of d_j with $\delta_j^{\widetilde{K}, \widetilde{W}}$ denoting the adjoint of the coboundary $d_j^{\widetilde{K}}$ with respect to the inner product on $C_{(2)}(\widetilde{K})$. Define the analytic Laplacian Δ_j to be equal to $d\delta + \delta d$ acting on $\Omega_{(2)}(\widetilde{M})$. We also define the combinatorial Laplacian $\Delta_j^{\widetilde{K}, \widetilde{W}}$ to be equal to $d^K \delta^{\widetilde{K}, \widetilde{W}} + \delta^{\widetilde{K}, \widetilde{W}} d^K$ acting on $C_{(2)}(\widetilde{K})$. Regard d_j as an operator on smooth j -forms in $(\ker d_j)^\perp$. Then we may define the spectral resolution for $D_j \equiv \delta_j d_j = \int_0^\infty \lambda dE_\lambda^j$. By results of Atiyah and Singer [2, 21] the spectral projections E_λ^j are in the commutant of the \mathcal{U} action and have finite von Neumann trace. So we can also define the spectral density function $F_j(\lambda) = \tau(E_\lambda^j)$. Similar comments apply to the combinatorial operator

$\delta_j^{\widetilde{K}, \widetilde{W}} d_j^{\widetilde{K}}$ so that we may similarly define its spectral projections $E_\lambda^{\widetilde{K}, \widetilde{W}, j}$ and the spectral density function $F_j^{\widetilde{K}, \widetilde{W}}(\lambda)$.

As in Dodziuk-Patodi [8], we make the following assumptions on our triangulations. Firstly, we consider only triangulations K which are rectilinear subdivisions of any fixed triangulation K_0 . The next assumption is that the fullness of K (see [25] p. 125 for the definition) is bounded away from zero.

We will next state a version of an elementary lemma which is due to Novikov and Shubin and whose proof can be found in the appendix of [12].

Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying the following sub-exponential estimate, $F(\lambda) = O(e^{\varepsilon\lambda})$ for all $\varepsilon > 0$. Then the Laplace transform of F exists, and is given by

$$\theta(t) = \int_0^\infty e^{-\lambda t} dF(\lambda).$$

Let $\bar{b} = \lim_{\varepsilon \rightarrow 0^+} F(\varepsilon) = F(0^+)$. Then $\bar{b} = \lim_{t \rightarrow \infty} \theta(t)$.

Lemma 1.1 : [12] *Let F and θ be as above, and let $\alpha > 0$. Then the following conditions are equivalent.*

(1). *There exists $C > 0$ such that*

$$C^{-1}\lambda^\alpha \leq F(\lambda) - \bar{b} \leq C\lambda^\alpha$$

for all λ small, $\lambda > 0$.

(2). *There exists $C' > 0$ such that*

$$C'^{-1}t^{-\alpha} \leq \theta(t) - \bar{b} \leq C't^{-\alpha}$$

for all $t \gg 0$.

More generally, one has the following equality

$$\liminf_{\lambda \rightarrow 0^+} \left\{ \frac{\log(F(\lambda) - \bar{b})}{\log \lambda} \right\} = \liminf_{t \rightarrow \infty} \left\{ - \frac{\log(\theta(t) - \bar{b})}{\log t} \right\}.$$

In our context we need to consider a slightly different situation. Let $G : \mathbb{R} \rightarrow \mathbb{R}_+$ be a non-decreasing function satisfying the sub-exponential estimate $G(\lambda) = O(e^{\varepsilon\lambda})$ for all $\varepsilon > 0$, and $G(\lambda) = 0$ for $\lambda < \lambda_0$ where $\lambda_0 > 0$. Let $\psi(t)$ denote its Laplace transform, that is,

$$\psi(t) = \int_{\lambda_0}^\infty e^{-\lambda t} dG(\lambda) \quad \text{for } t > 0.$$

Let $\bar{a} = \lim_{\lambda \rightarrow \lambda_0^+} G(\lambda) = \lim_{t \rightarrow \infty} \psi(t)$. Then the following lemma is an easy consequence of Lemma 1.1.

Lemma 1.2 : *Let G and ψ be as above, and let $\alpha > 0$. Then the following conditions are equivalent.*

(1). *There exists $C > 0$ such that*

$$C^{-1}(\lambda - \lambda_0)^\alpha \leq G(\lambda) - \bar{a} \leq C(\lambda - \lambda_0)^\alpha$$

for all $\lambda > \lambda_0$ and $\lambda - \lambda_0$ small.

(2). *There exists $C' > 0$ such that*

$$C'^{-1}e^{-\lambda_0 t}t^{-\alpha} \leq \psi(t) - \bar{a} \leq C'e^{-\lambda_0 t}t^{-\alpha}$$

for all $t \gg 0$.

More generally, one has the following equality

$$\lim_{\lambda \rightarrow \lambda_0^+} \inf_{\lambda - \lambda_0 \rightarrow 0^+} \left\{ \frac{\log(G(\lambda) - \bar{a})}{\log(\lambda - \lambda_0)} \right\} = \lim_{t \rightarrow \infty} \inf_{t \rightarrow \infty} \left\{ -\frac{\log(\psi(t) - \bar{a})}{\log t} - \frac{\lambda_0 t}{\log t} \right\}.$$

In fact, we shall use below the one-sided versions of the above Lemma (for example, the RHS inequality in (1) is equivalent to the RHS inequality in (2)), and also the fact that the constants in (1) and (2) are connected: if the situation depends on some parameter, C and C' go to 0 or $+\infty$ at the same time.

2 The Convergence of Spectral Functions

In this section, we prove that the von Neumann spectral density of the combinatorial Laplacian converges to the von Neumann spectral density of the analytic Laplacian as we successively refine our triangulations so that the mesh goes to zero. We also prove the analogous theorem for the bottom of the spectra of the combinatorial and analytic Laplacians. We use the notation $H^s(\widetilde{M})$ for the Sobolev space consisting of forms ω with

$$\int_{\widetilde{M}} \|(1 + \Delta)^{s/2} \omega\| < \infty. \quad (2.1)$$

For the readers' convenience we now recall (cf [7]) the definitions of the de Rham and Whitney maps. For every continuous form ω of degree p on \widetilde{M} we define a cochain $\int \omega \in C^p(\widetilde{K})$ by the formula

$$\int \omega = \sum_{\sigma} \left(\int_{\sigma} \omega \right) \cdot \sigma.$$

Lemma 2.1 : [7] *Suppose $\omega \in H^k(\widetilde{M})$, $k > \frac{n}{2} + 1$, has degree p . Define the de Rham map $\widetilde{A}\omega = \int \omega$, then $\widetilde{A}\omega$ is in $C_{(2)}(\widetilde{K})$. Moreover $\widetilde{A} : H^k(\widetilde{M}) \rightarrow C_{(2)}(\widetilde{K})$ is bounded and $\widetilde{A}d = d^{\widetilde{K}}\widetilde{A}$.*

Let $\{U_p\}_{p \in K^0}$ be an open covering of M by open stars of vertices of K , that is, $U_v = \text{st}(v)$ for $v \in K^0$. There is a partition of unity on M subordinate to this covering and which is given by the barycentric

coordinate functions. Both the covering and the partition of unity can be lifted to \widetilde{M} so that we obtain a partition of unity $\{\mu_v\}_{v \in \widetilde{K}^0}$ indexed by vertices of \widetilde{K} with the following properties:

$$\begin{aligned} \text{supp } \mu_v &\subset \text{supp st } v \\ \mu_v \cdot \gamma &= \mu_{\gamma^{-1}v} . \end{aligned} \tag{2.2}$$

Now let $\sigma = [v_0, \dots, v_p]$ be a simplex of K . To simplify the notation we write $\mu_i = \mu_{v_i}$. Define the Whitney map

$$\widetilde{W}\sigma = \begin{cases} \mu_0 & \text{if } p = 0 \\ p! \sum_{i=0}^p (-1)^i \mu_i d\mu_0 \wedge \dots \wedge d\mu_{i-1} \wedge d\mu_{i+1} \wedge \dots \wedge d\mu_p & \text{if } p > 0. \end{cases} \tag{2.3}$$

Note that $\gamma^* \widetilde{W}\sigma = \widetilde{W}\gamma^{-1}\sigma$ by (2.2). Another property of $\widetilde{W}\sigma$ which we need is

$$\text{supp } \widetilde{W}\sigma = \text{supp st } \sigma. \tag{2.4}$$

Now for an arbitrary cochain $f = \sum f_\sigma c_\sigma$ we define $\widetilde{W}f = \sum f_\sigma \widetilde{W}\sigma$. In view of (2.4), the sum in (2.3) is locally finite and defines an L^2 form. For every $f \in C_{(2)}(\widetilde{K})$ the form $\widetilde{W}f$ is in $\Omega_{(2)}(\widetilde{M})$. Moreover $\widetilde{W} : C_{(2)}(\widetilde{K}) \rightarrow \Omega_{(2)}(\widetilde{M})$ is a bounded operator with the following properties.

$$\begin{aligned} \text{(i)} \quad & d \cdot \widetilde{W} = \widetilde{W} \cdot d^{\widetilde{K}} \\ \text{(ii)} \quad & \widetilde{A}\widetilde{W} = I \quad \text{on } C_{(2)}(\widetilde{K}) \\ \text{(iii)} \quad & \gamma^* \widetilde{W}f = \widetilde{W}(f \cdot \gamma) \quad \text{for every } f \in C_{(2)}(\widetilde{K}) \text{ and every } \gamma \in \Gamma. \end{aligned} \tag{2.5}$$

The first two properties are provided in [25]. The third one is a consequence of our choice of partition of unity.

Now we review some of the results of [10] and [12] which we need. Let H denote a Hilbert space with a trivial action of Γ and let $V \subseteq \ell^2(\Gamma) \otimes H$ be a closed Γ invariant subspace. Let P_V denote the orthogonal projection onto V . Then we recall that $\dim_\Gamma(V) = \tau(P_V)$. It can be seen that $\dim_\Gamma(V)$ is independent of the choice of the Γ -invariant embedding of V as a subspace of $\ell^2(\Gamma) \otimes HS$, where HS denotes a Hilbert space with a trivial action of Γ . Let \mathcal{S}_λ^j denote the set of Γ invariant subspaces L of $(\ker d_j)^\perp \subseteq \Omega_{(2)}^j(\widetilde{M})$ such that

$$\|d_j \omega\| \leq \sqrt{\lambda} \|\omega\| \quad \text{for all } \omega \in L \tag{2.6}$$

and $\mathcal{S}_\lambda^{j, \widetilde{K}, \widetilde{W}}$ denote the set of Γ invariant subspaces L of $(\ker d_j^{\widetilde{K}})^\perp \subseteq C_{(2)}(\widetilde{K})$ such that

$$\|d_j^{\widetilde{K}} a\|_{\widetilde{W}} \leq \sqrt{\lambda} \|a\|_{\widetilde{W}} \quad \forall a \in L. \tag{2.7}$$

Then the respective spectral functions are also given by

$$F_j(\lambda) \equiv \sup\{\dim_\Gamma(L) \mid L \in \mathcal{S}_\lambda^j\} \quad (2.8)$$

and

$$F_j^{\tilde{K}, \tilde{W}}(\lambda) = \sup\{\dim_\Gamma(L) \mid L \in \mathcal{S}_\lambda^{j, \tilde{K}, \tilde{W}}\} \quad (2.9)$$

by the variational principle of [10] and [12]. This principle is the analogue of the min-max principle in finite dimensions, and gives a variational characterisation of the von Neumann spectrum of our Laplace operators. We introduce the notation η for the mesh of the triangulation \tilde{K} (which is of course the mesh for K).

Our first main result in this section is as follows,

Proposition 2.2 : *There is a constant C_1 independent of η such that if $\mu < \left(\frac{1}{C_1\eta|\log \eta|} - 1\right)^2$ and $\lambda \leq \mu$, then*

$$F_j^{\tilde{K}, \tilde{W}}(\lambda) \leq F_j(D_\eta^\mu \lambda),$$

where $D_\eta^\mu = \left(1 - C_1\eta|\log \eta|(1 + \sqrt{\mu})\right)^{-2} \rightarrow 1$ as μ is fixed and $\eta \rightarrow 0$.

The proof is contained in the following sequence of three lemmas. Let $P, P^{\tilde{K}}$ denote the projections onto $(\ker d_j)^\perp$ and $(\ker d_j^{\tilde{K}})^\perp$ respectively.

Lemma 2.3 : *$P\tilde{W}$ is injective on L , where L is a Γ -invariant subspace of $(\ker d_j^{\tilde{K}})^\perp$.*

Proof: Let $a \in L$ and $P\tilde{W}a = 0$, that is, $\tilde{W}a \in \ker d_j$. Then

$$\tilde{W}d_j^{\tilde{K}}a = d_j\tilde{W}a = 0.$$

Hence $d_j^{\tilde{K}}a = 0$, since \tilde{W} is injective. But $a \in (\ker d_j^{\tilde{K}})^\perp$, so $a = 0$.

An immediate consequence is that

$$\dim_\Gamma(L) = \dim_\Gamma(P\tilde{W}(L)).$$

This is because $P\tilde{W}$ is injective on such subspaces and injective maps of \mathcal{U} -modules into free modules preserve the Γ dimension of the space.

Recall that $E_\mu^{\tilde{K}, \tilde{W}}C_{(2)}^j(\tilde{K}) = \left\{a \in C_{(2)}^j(\tilde{K}) : \|d_j^{\tilde{K}}a\|_{\tilde{W}}^2 \leq \mu\|a\|_{\tilde{W}}^2\right\}$.

The following is a highly non-trivial estimate in [8].

Lemma 2.4 : *Let $a \in E_\mu^{\tilde{K}, \tilde{W}}C_{(2)}^j(\tilde{K})$, then one has for η sufficiently small*

$$\|\tilde{W}P^{\tilde{K}}a - P\tilde{W}a\| \leq C_1(1 + \sqrt{\mu})\eta|\log \eta| \|a\|_{\tilde{W}}.$$

Lemma 2.5 : If $a \in E_{\mu}^{\tilde{K}, \tilde{W}} C_{(2)}^j(\tilde{K})$ satisfies $\|d^{\tilde{K}} a\|_{\tilde{W}} \leq \sqrt{\lambda} \|a\|_{\tilde{W}}$ and $a \in (\ker d_j^{\tilde{K}})^{\perp}$, then one has

$$\|dP\tilde{W}a\| \leq \left(1 - C_1\eta |\log \eta| (1 + \sqrt{\mu})\right)^{-1} \sqrt{\lambda} \|P\tilde{W}a\|,$$

for η sufficiently small with respect to μ .

Proof: Since $d(1 - P) = 0$, we see that

$$\begin{aligned} \|dP\tilde{W}a\| &= \|\tilde{W}d^{\tilde{K}}a\| = \|d^{\tilde{K}}a\|_{\tilde{W}} \\ &\leq \sqrt{\lambda} \|a\|_{\tilde{W}} = \sqrt{\lambda} \|P^{\tilde{K}}a\|_{\tilde{W}} \end{aligned}$$

since by hypothesis $a = P^{\tilde{K}}a$.

Lemma 2.4 and the equality $a = P^{\tilde{K}}a$ imply that

$$\|P^{\tilde{K}}a\|_{\tilde{W}} \leq \left(1 - C_1\eta |\log \eta| (1 + \sqrt{\mu})\right)^{-1} \|P\tilde{W}a\|.$$

This proves the lemma.

Proof of Proposition 2.2: For fixed μ and for η sufficiently small, let $D_{\eta}^{\mu} = \left(1 - C_1\eta |\log \eta| (1 + \sqrt{\mu})\right)^{-2}$.

It follows from Lemma 2.5 that if $L \in \mathcal{S}_{\lambda}^{j, \tilde{K}, \tilde{W}}$ and $\lambda \leq \mu$, then $P\tilde{W}(L) \in S_{D_{\eta}^{\mu}\lambda}^j$, and from Lemma 2.3 that $\dim_{\Gamma}(L) = \dim_{\Gamma}(P\tilde{W}(L))$.

The proposition is then merely a consequence of the definition of F_j and $F_j^{\tilde{K}, \tilde{W}}$.

Our next main result of this section is as follows,

Proposition 2.6 : There is a constant C_2 independent of η such that if $\mu < (C_2\eta)^{-2/s} - 1$ and $\lambda \leq \mu$, then $F_j(\lambda) \leq F_j^{\tilde{K}, \tilde{W}}(C_{\eta}^{\mu}\lambda)$, where $C_{\eta}^{\mu} = \left\{\frac{1 + C_2\eta(1 + \mu)^{\frac{s}{2}}}{1 - C_2\eta(1 + \mu)^{\frac{s}{2}}}\right\}^2 \rightarrow 1$ as $\eta \rightarrow 0$.

The proof relies on the following sequence of lemmas.

Lemma 2.7 [7, 8] The inequality $\|\omega - \tilde{W}\tilde{A}\omega\| \leq C_2\eta\|\omega\|_s$ holds for all $\omega \in H^s(\tilde{M})$ and $s > \frac{n}{2} + 1$ where C_2 is independent of η .

Recall that $E_{\mu}\Omega_{(2)}^j(\tilde{M}) = \left\{\omega \in \Omega_{(2)}^j(\tilde{M}) : \|d_j\omega\|^2 \leq \mu\|\omega\|^2\right\}$.

The following lemma is taken from [9], and its proof is easy.

Lemma 2.8 : For $\omega \in E_{\mu}\Omega_{(2)}^j(\tilde{M})$ and $s \geq 0$,

$$\|\omega\|^2 \leq \|\omega\|_s^2 \leq (1 + \mu)^s \|\omega\|^2.$$

It follows from this lemma that the de Rham map is well defined and bounded on the range of E_μ .

From now on, we shall fix $s > \frac{n}{2} + 1$.

Lemma 2.9 [9]: For $\omega \in E_\mu \Omega_{(2)}^j(\widetilde{M}) \cap (\ker d_j)^\perp$ and η sufficiently small, one has

$$\|\omega\| \leq \frac{1}{1 - C_2 \eta (1 + \mu)^{\frac{s}{2}}} \|P^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}}.$$

Proof: The operator

$$B = (1 - \widetilde{W} \widetilde{A}) E_\mu : \Omega_{(2)}^j(\widetilde{M}) \rightarrow \Omega_{(2)}^j(\widetilde{M})$$

is bounded for all j with $\|B\| < C_2 \eta (1 + \mu)^{\frac{s}{2}}$ (using Lemmas 2.7 and 2.8). Now because $\omega = \widetilde{W} \widetilde{A} \omega + B \omega$, and also as $\widetilde{W} : \ker d_j^{\widetilde{K}} \rightarrow \ker d_j$ and $\widetilde{A} : \ker d_j \rightarrow \ker d_j^{\widetilde{K}}$, we conclude that

$$\begin{aligned} \|\omega\| &= \|P \omega\| \leq \|P \widetilde{W} \widetilde{A} \omega\| + \|P B \omega\| \\ &\leq \|P \widetilde{W} P^{\widetilde{K}} \widetilde{A} \omega\| + \|B \omega\| \\ &\leq \|P^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}} + C_2 \eta (1 + \mu)^{\frac{s}{2}} \|\omega\|. \end{aligned}$$

An immediate consequence of this result is that the map $P^{\widetilde{K}} \widetilde{A}$ is injective on L , where L is a Γ invariant subspace of $E_\mu \Omega_{(2)}^j(\widetilde{M}) \cap (\ker d_j)^\perp$ as soon as

$$\eta < \frac{1}{C_2 (1 + \mu)^{\frac{s}{2}}}.$$

Hence, for η small enough with respect to μ ,

$$\dim_\Gamma(L) = \dim_\Gamma(P^{\widetilde{K}} \widetilde{A} L)$$

for all Γ -invariant subspaces L in the range of E_λ where $\lambda \leq \mu$. Again, this is because injective maps of \mathcal{U} -modules into free modules preserve the Γ dimension of the space.

Lemma 2.10 : Let $\omega \in E_\mu \Omega_{(2)}^j(\widetilde{M})$ satisfy $\|d\omega\| \leq \sqrt{\lambda} \|\omega\|$ and $\omega \in (\ker d_j)^\perp$. Then, for η small enough,

$$\|d^{\widetilde{K}} P^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}} \leq \left\{ \frac{1 + C_2 \eta (1 + \mu)^{\frac{s}{2}}}{1 - C_2 \eta (1 + \mu)^{\frac{s}{2}}} \right\} \sqrt{\lambda} \|P^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}}.$$

Proof: Notice first that since $d^{\widetilde{K}}(I - P^{\widetilde{K}}) = 0$, one has

$$\|d^{\widetilde{K}} P^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}} = \|d^{\widetilde{K}} \widetilde{A} \omega\|_{\widetilde{W}} = \|\widetilde{W} d^{\widetilde{K}} \widetilde{A} \omega\| = \|\widetilde{W} \widetilde{A} d\omega\|.$$

Now $d\omega \in E_\mu \Omega_{(2)}^{j+1}(\widetilde{M})$, since $d^2 \omega = 0$. Therefore according to Lemmas 2.8 and 2.9,

$$\|d\omega - \widetilde{W} \widetilde{A} d\omega\| \leq C_2 \eta (1 + \mu)^{\frac{s}{2}} \|d\omega\|.$$

Therefore using the hypothesis $\|d\omega\| \leq \sqrt{\lambda}\|\omega\|$, one has

$$\|\widetilde{W}\widetilde{A}d\omega\| \leq (1 + C_2\eta(1 + \mu)^{\frac{s}{2}})\sqrt{\lambda}\|\omega\|.$$

The conclusion of the lemma follows from Lemma 2.9.

Proof of Proposition 2.6: For fixed μ and for η sufficiently small, let $C_\eta^\mu = \left\{ \frac{1 + C_2\eta(1 + \mu)^{\frac{s}{2}}}{1 - C_2\eta(1 + \mu)^{\frac{s}{2}}} \right\}^2$.

It follows from Lemma 2.10 that if $L \in S_\lambda^j$ and $\lambda \leq \mu$, then $P^{\widetilde{K}}\widetilde{A}(L) \in \mathcal{S}_{C_\eta^\mu\lambda}^{j,\widetilde{K},\widetilde{W}}$, and from Lemma 2.9 that $\dim_\Gamma(L) = \dim_\Gamma(P^{\widetilde{K}}\widetilde{A}L)$.

The proposition is then merely a consequence of the definition of F_j and $F_j^{\widetilde{K},\widetilde{W}}$.

We will now prove that $\kappa_{0,j}^{\widetilde{K},\widetilde{W}}$, the bottom of the spectrum of $\delta_j^{\widetilde{K},\widetilde{W}}d_j^{\widetilde{K}}$ acting on $\delta_j^{\widetilde{K},\widetilde{W}}C_{(2)}^{j+1}(\widetilde{K})$, converges to $\kappa_{0,j}$, the bottom of the spectrum of $\delta_j d_j$ acting on $\delta_j \Omega_{(2)}^{j+1}(\widetilde{M})$, as the mesh of the triangulation K goes to zero.

First notice that $\overline{\delta_j^{\widetilde{K},\widetilde{W}}C_{(2)}^{j+1}(\widetilde{K})} = (\ker d_j^{\widetilde{K}})^\perp = (\ker \delta_j^{\widetilde{W},\widetilde{K}}d_j^{\widetilde{K}})^\perp$, and that $\overline{\delta_j \Omega_{(2)}^{j+1}(\widetilde{M})} = (\ker d_j)^\perp = (\ker \delta_j d_j)^\perp$.

Therefore

$$\begin{aligned} \kappa_{0,j}^{\widetilde{K},\widetilde{W}} &= \inf \left\{ \frac{\|d_j^{\widetilde{K}}a\|_{\widetilde{W}}^2}{\|a\|_{\widetilde{W}}^2} : a \in (\ker d_j^{\widetilde{K}})^\perp \right\} \\ \text{and } \kappa_{0,j} &= \inf \left\{ \frac{\|d\omega\|^2}{\|\omega\|^2} : \omega \in (\ker d_j)^\perp \right\}. \end{aligned}$$

Theorem 2.11 : *The following inequality holds*

$$D_\eta^{-1}\kappa_{0,j} \leq \kappa_{0,j}^{\widetilde{K},\widetilde{W}} \leq C_\eta\kappa_{0,j},$$

where D_η and C_η tend to 1^+ as η goes to zero.

In particular, $\kappa_{0,j}^{\widetilde{K},\widetilde{W}}$ converges to $\kappa_{0,j}$ as the mesh η of the triangulation K goes to zero.

Proof: Take $\varepsilon > 0$, and let $\omega \in (\ker d_j)^\perp$ be such that

$$\|d\omega\| \leq (\kappa_{0,j} + \varepsilon)^{1/2}\|\omega\|.$$

According to Lemma 2.10, for η small enough,

$$\|d_j^{\widetilde{K}}P^{\widetilde{K}}\widetilde{A}\omega\|_{\widetilde{W}} \leq \left\{ \frac{1 + C_2\eta(1 + \kappa_{0,j} + \varepsilon)^{\frac{s}{2}}}{1 - C_2\eta(1 + \kappa_{0,j} + \varepsilon)^{\frac{s}{2}}} \right\} \sqrt{\kappa_{0,j} + \varepsilon} \|P^{\widetilde{K}}\widetilde{A}\omega\|_{\widetilde{W}}.$$

Therefore

$$\kappa_{0,j}^{\widetilde{K},\widetilde{W}} \leq \left\{ \frac{1 + C_2\eta(1 + \kappa_{0,j} + \varepsilon)^{\frac{s}{2}}}{1 - C_2\eta(1 + \kappa_{0,j} + \varepsilon)^{\frac{s}{2}}} \right\}^2 (\kappa_{0,j} + \varepsilon).$$

Since ε is arbitrary, this gives

$$\kappa_{0,j}^{\tilde{K},\tilde{W}} \leq \left\{ \frac{1 + C_2\eta(1 + \kappa_{0,j})^{\frac{\varepsilon}{2}}}{1 - C_2\eta(1 + \kappa_{0,j})^{\frac{\varepsilon}{2}}} \right\}^2 \kappa_{0,j},$$

and one sets

$$C_\eta = \left\{ \frac{1 + C_2\eta(1 + \kappa_{0,j})^{\frac{\varepsilon}{2}}}{1 - C_2\eta(1 + \kappa_{0,j})^{\frac{\varepsilon}{2}}} \right\}^2.$$

Take $\varepsilon > 0$, and let $a \in (\ker d_j^{\tilde{K}})^\perp$ be such that

$$\|d_j^{\tilde{K}} a\|_{\tilde{W}} \leq (\kappa_{0,j}^{\tilde{K},\tilde{W}} + \varepsilon)^{1/2} \|a\|_{\tilde{W}}.$$

For η small enough, $(1 - C_1\eta|\log \eta|(1 + (\kappa_{0,j}^{\tilde{K},\tilde{W}} + \varepsilon)^{1/2}))$ is positive, since by the inequality just proven $\kappa_{0,j}^{\tilde{K},\tilde{W}}$ is bounded when η goes to zero. According to Lemma 2.5,

$$\|dP\tilde{W}a\| \leq \left(1 - C_1\eta|\log \eta|(1 + (\kappa_{0,j}^{\tilde{K},\tilde{W}} + \varepsilon)^{1/2})\right)^{-1} (\kappa_{0,j}^{\tilde{K},\tilde{W}} + \varepsilon)^{1/2} \|P\tilde{W}a\|,$$

hence

$$\kappa_{0,j} \leq \left(1 - C_1\eta|\log \eta|(1 + \sqrt{\kappa_{0,j}^{\tilde{K},\tilde{W}}})\right)^{-2} \kappa_{0,j}^{\tilde{K},\tilde{W}}.$$

Now, if $\kappa_{0,j}^{\tilde{K},\tilde{W}}$ is bounded from above by K as η goes to zero, one sets

$$D_\eta = \left(1 - C_1\eta|\log \eta|(1 + \sqrt{K})\right)^{-2}.$$

According to [8], p.11 (see also [12] p.385), the spectrum of $\delta_j d_j$ acting on $\delta_j \Omega_{(2)}^{j+1}(\tilde{M})$ coincides with the spectrum of $d_j \delta_j$ acting on $d_j \Omega_{(2)}^j(\tilde{M})$. Therefore one has

$$\lambda_{0,j} = \text{minimum}\{\kappa_{0,j-1}, \kappa_{0,j}\},$$

where $\lambda_{0,j}$ denotes the bottom of the spectrum of the Laplacian Δ_j acting on $\Omega_{(2)}^j(\tilde{M})$. Similar remarks apply in the combinatorial situation, and we can therefore state

Corollary 2.12 : *The bottom of the spectrum of the combinatorial Laplacian $\Delta_j^{\tilde{K},\tilde{W}}$ acting on $C_{(2)}^j(\tilde{K})$, $\lambda_{0,j}^{\tilde{K},\tilde{W}}$, converges to the bottom of the spectrum of the Laplacian Δ_j acting on $\Omega_{(2)}^j(\tilde{M})$, $\lambda_{0,j}$. More precisely, the following inequality holds*

$$D_\eta^{-1} \lambda_{0,j} \leq \lambda_{0,j}^{\tilde{K},\tilde{W}} \leq C_\eta \lambda_{0,j}.$$

where D_η and C_η tend to 1^+ as η goes to zero.

3 Estimates for L^2 Theta Functions.

In this section, we prove some basic estimates for the L^2 theta functions which we will need later on in the paper.

Note firstly that if $\bar{N}_j = \tau(E_{\Delta_j})$, where $\Delta_j = \int \lambda dE_{\Delta_j}(\lambda)$ is the spectral resolution of the Laplacian Δ_j , then $\bar{N}_j = F_{j-1} + F_j + b_{(2)}^j(\bar{M})$ (cf. [12]). One sets $F_{-1} = F_{n+1} = 0$. Let $\bar{N}_j = N_j + b_{(2)}^j(\bar{M})$, that is, $N_j = F_{j-1} + F_j$. Recall the definition of the theta function for the Laplacian on L^2 j -forms:

$$\theta_j(t) = \tau(e^{-t\Delta_j}) - b_{(2)}^j(\bar{M}) = t \int_0^\infty e^{-t\lambda} N_j(\lambda) d\lambda.$$

Since

$$\frac{d}{dt} \theta_j(t) = -\tau(\Delta_j e^{-t\Delta_j}) < 0,$$

θ_j is a decreasing function.

We use a similar notation for the theta function of the combinatorial Laplacian, that is,

$$\theta_j^{\tilde{K}, \tilde{W}}(t) = t \int_0^\infty e^{-t\lambda} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda.$$

For the needs of section 4, we consider the truncated theta functions

$$\theta_{j,\nu}(t) = t \int_{\lambda_{0,j} + \nu_\eta}^\infty e^{-\lambda t} N_j(\lambda) d\lambda.$$

and

$$\theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) = t \int_{\lambda_{0,j} + \nu_\eta}^\infty e^{-\lambda t} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda$$

where ν_η is non-negative and bounded. Our aim is to investigate the convergence of the former to the latter as the mesh goes to zero. Note that $\theta_j^{\tilde{K}, \tilde{W}}$, $\theta_{j,\nu}$, $\theta_{j,\nu}^{\tilde{K}, \tilde{W}}$ are decreasing as well.

Proposition 3.1 *There is a $t_0 = t_0(\eta)$ such that if $t \geq t_0 > 0$, then*

$$(1). \quad e^{\lambda_{0,j} D_\eta^{-1} t} \theta_{j,\nu}(C'_\eta t) \leq e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) + \varepsilon(\eta)$$

$$(2). \quad e^{\lambda_{0,j} D_\eta^{-1} t} \theta_j(C'_\eta t) \leq e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t) + \varepsilon(\eta)$$

$$(3). \quad \theta_j(C'_\eta t) \leq \theta_j^{\tilde{K}, \tilde{W}}(t) + \varepsilon(\eta)$$

where $t_0 \rightarrow 0$, $\varepsilon \rightarrow 0$ as $\eta \rightarrow 0$, D_η is as in Proposition 2.12 and C'_η tends to 1^+ as η goes to zero.

Proof: Choose $\mu = \eta^{-\alpha}$, $0 < \alpha < 2/s$. Then, for η small enough, C_η^μ exists and tends to one as η goes to zero. Define $C'_\eta = C_\eta^\mu$. Write

$$e^{\lambda_{0,j} D_\eta^{-1} t} \theta_{j,\nu}(C'_\eta t) = C'_\eta t e^{\lambda_{0,j} D_\eta^{-1} t} \int_{\lambda_{0,j} + \nu_\eta}^\mu e^{-t C'_\eta \lambda} N_j(\lambda) d\lambda + C'_\eta t e^{\lambda_{0,j} D_\eta^{-1} t} \int_\mu^\infty e^{-t C'_\eta \lambda} N_j(\lambda) d\lambda.$$

By Propositions 2.6 and 2.12,

$$\begin{aligned} C'_\eta t e^{\lambda_{0,j} D_\eta^{-1} t} \int_{\lambda_{0,j} + \nu_\eta}^\mu e^{-t C'_\eta \lambda} N_j(\lambda) d\lambda &\leq C'_\eta t e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \int_{\lambda_{0,j} + \nu_\eta}^\mu e^{-t C'_\eta \lambda} N_j^{\tilde{K}, \tilde{W}}(C'_\eta \lambda) d\lambda \\ &\leq e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t). \end{aligned}$$

We now proceed to estimate the error

$$\varepsilon(\eta, t) \equiv C'_\eta t \int_\mu^\infty e^{-t(C'_\eta \lambda - D_\eta^{-1} \lambda_{0,j})} N_j(\lambda) d\lambda.$$

Recall that the small time asymptotics of the heat kernel on differential forms on the covering space is given for $n = \dim M$ by

$$\tau(e^{-t\Delta_j}) = c_j t^{-n/2} + R(t)$$

where $\lim_{t \rightarrow 0} t^{n/2} R(t) = 0$ as in [20]. Hence by the Tauberian theorem, the function $N_j(\lambda)$ has the large λ asymptotic expansion

$$N_j(\lambda) = c_j \lambda^{n/2} + f_j(\lambda)$$

where $\lim_{\lambda \rightarrow \infty} \lambda^{-n/2} f_j(\lambda) = 0$. It follows that, given $\epsilon > 0$, there is a $\Lambda_j > 0$ such that for all $\lambda > \Lambda_j$

$$N_j(\lambda) \leq (c_j + \epsilon) \lambda^{n/2}.$$

Hence, if $t \geq t_0 = \frac{1}{\sqrt{C'_\eta \mu - D_\eta^{-1} \lambda_{0,j}}}$, then $\varepsilon(\eta, t) \leq \varepsilon(\eta, t_0)$. Now, μ tends to infinity as η goes to zero. Thus, for η small enough,

$$\begin{aligned} \varepsilon(\eta, t_0) &\leq C'_\eta t_0 (c_j + \epsilon) \int_\mu^\infty e^{-t_0(C'_\eta \lambda - D_\eta^{-1} \lambda_{0,j})} \lambda^{n/2} d\lambda \\ &\leq C'_\eta t_0 (c_j + \epsilon) e^{-\frac{t_0}{2}(C'_\eta \mu - D_\eta^{-1} \lambda_{0,j})} \int_\mu^\infty e^{-\frac{t_0}{2}(C'_\eta \lambda - D_\eta^{-1} \lambda_{0,j})} \lambda^{n/2} d\lambda \\ &\leq c'_j (C'_\eta)^{-n/2} (C'_\eta \mu - D_\eta^{-1} \lambda_{0,j})^{\frac{n}{4}} e^{-\frac{\sqrt{C'_\eta \mu - D_\eta^{-1} \lambda_{0,j}}}{2}} \rightarrow 0 \quad \text{as } \eta \rightarrow 0 \end{aligned}$$

where

$$c'_j = (c_j + \epsilon) 2^{\frac{n}{2}+1} \int_0^\infty e^{-y} (y + \lambda_{0,j})^{\frac{n}{2}} dy.$$

The proof above also works for parts (2) and (3).

As a converse to the previous result we establish the following inequality.

Proposition 3.2 *There is a $t_0 = t_0(\eta)$ such that if $t \geq t_0 > 0$, then*

$$(1). \quad e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) \leq e^{\lambda_{0,j} C_\eta t} \theta_{j,\nu}(D_\eta'^{-1} t) + \varepsilon(\eta)$$

$$(2). \quad e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t) \leq e^{\lambda_{0,j} C_\eta t} \theta_j(D_\eta'^{-1} t) + \varepsilon(\eta)$$

$$(3). \quad \theta_j^{\tilde{K}, \tilde{W}}(t) \leq \theta_j(D_\eta'^{-1} t) + \varepsilon(\eta)$$

where $t_0 \rightarrow 0$, $\varepsilon \rightarrow 0$ as η goes to zero, C_η is as in Proposition 2.12 and D'_η tends to 1^+ as η goes to zero.

Proof: Choose for example $\mu = \eta^{-\alpha}$, $0 < \alpha < 1$. Then, for η small enough, D_η^μ exists and tends to one as η goes to zero. Define $D'_\eta = D_\eta^\mu$. Write

$$e^{\lambda_{0,j}^{\tilde{K},\tilde{W}} t} \theta_{j,\nu}^{\tilde{K},\tilde{W}}(t) = e^{\lambda_{0,j}^{\tilde{K},\tilde{W}} t} \int_{\lambda_{0,j} + \nu_\eta}^{\mu} e^{-t\lambda} N_j^{\tilde{K},\tilde{W}}(\lambda) d\lambda + e^{\lambda_{0,j}^{\tilde{K},\tilde{W}} t} \int_{\mu}^{\infty} e^{-t\lambda} N_j^{\tilde{K},\tilde{W}}(\lambda) d\lambda.$$

By Propositions 2.2 and 2.12,

$$\begin{aligned} e^{\lambda_{0,j}^{\tilde{K},\tilde{W}} t} \int_{\lambda_{0,j} + \nu_\eta}^{\mu} e^{-t\lambda} N_j^{\tilde{K},\tilde{W}}(\lambda) d\lambda &\leq e^{\lambda_{0,j}^{\tilde{K},\tilde{W}} t} \int_{\lambda_{0,j} + \nu_\eta}^{\mu} e^{-t\lambda} N_j(D'_\eta \lambda) d\lambda \\ &\leq e^{\lambda_{0,j} C_\eta t} \theta_{j,\nu}(D_\eta'^{-1} t). \end{aligned}$$

We now proceed to estimate the error

$$\varepsilon(\eta, t) \equiv t \int_{\mu}^{\infty} e^{-t(\lambda - \lambda_{0,j}^{\tilde{K},\tilde{W}})} N_j^{\tilde{K},\tilde{W}}(\lambda) d\lambda.$$

By the fullness assumption on the triangulation \tilde{K} , $\dim_{\Gamma}(C_{(2)}^j(\tilde{K})) = \dim(C^j(K))$ grows like η^{-n} as $\eta \rightarrow 0$, see [8]. One then sees that the spectral density function of the combinatorial Laplacian on L^2 j -cochains $N_j^{\tilde{K},\tilde{W}}(\lambda)$ must be bounded,

$$N_j^{\tilde{K},\tilde{W}}(\lambda) \leq c_j \eta^{-n}.$$

Hence, if $t \geq t_0 = \frac{1}{\sqrt{\mu - \lambda_{0,j}^{\tilde{K},\tilde{W}}}}$, then $\varepsilon(\eta, t) \leq \varepsilon(\eta, t_0)$ and

$$\begin{aligned} \varepsilon(\eta, t_0) &\leq t_0 c_j \eta^{-n} \int_{\mu}^{\infty} e^{-t_0(\lambda - \lambda_{0,j}^{\tilde{K},\tilde{W}})} d\lambda \\ &= c_j \eta^{-n} e^{-t_0(\mu - \lambda_{0,j}^{\tilde{K},\tilde{W}})} \\ &= c_j \eta^{-n} e^{-\sqrt{\mu - \lambda_{0,j}^{\tilde{K},\tilde{W}}}} \rightarrow 0 \quad \text{as } \eta \rightarrow 0. \end{aligned}$$

The proof above also works for parts (2) and (3).

Finally, since θ_j , $\theta_j^{\tilde{K},\tilde{W}}$, $\theta_{j,\nu}$, and $\theta_{j,\nu}^{\tilde{K},\tilde{W}}$ are decreasing, one can replace C_η and C'_η by $\sup(C_\eta, C'_\eta)$, D_η and D'_η by $\sup(D_\eta, D'_\eta)$ in 2.12, 3.1, and 3.2; this means that from now on we can identify C_η and C'_η , D_η and D'_η , as long as we only use the fact that they tend to one from above as η goes to zero.

4 The Main Approximation Theorem

In this section, we prove our main approximation theorem, Theorem 4.1. This is proved using the approximation theorems for the truncated L^2 theta functions of the previous section, as well as the approximation theorem for the bottom of the spectra of Laplacians in section 2. We will assume that $\lambda_{0,j} > 0$ in this section. The case when $\lambda_{0,j} = 0$ is proved in section 7.

Recall the definition of the β invariant,

$$\begin{aligned}\beta_j(M, g) &= \sup\{\beta \in \mathbb{R} : e^{\lambda_{0,j}t} \theta_{j,\Delta}(t) \text{ is } O(t^{-\beta}) \text{ as } t \rightarrow \infty\} \in [0, \infty] \\ &= \liminf_{t \rightarrow \infty} \left\{ -\frac{\log \theta_j(t)}{\log t} - \frac{\lambda_{0,j}t}{\log t} \right\}\end{aligned}$$

Then our main theorem is

Theorem 4.1 : *Assume that $\beta_j(M, g) > 0$. Then in the limit as the mesh η of the triangulation K goes to zero, $e^{\lambda_{0,j}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t)$ converges uniformly to $e^{\lambda_{0,j}t} \theta_j(t)$ on $[t_0, +\infty[$ for any $t_0 > 0$.*

The proof of this theorem is an immediate consequence of the following three lemmas. For the rest of this section, we assume that $\beta_j(M, g) > 0$. First choose ν_η satisfying the following conditions:

- (1) $\nu_\eta > 0$ and $\nu_\eta \rightarrow 0$ as $\eta \rightarrow 0$
- (2) $\lambda_{0,j}^{\tilde{K}, \tilde{W}} < \lambda_{0,j} + \nu_\eta$
- (3) $D_\eta^{-1} \lambda - \lambda_{0,j} C_\eta > 0$ for $\lambda \geq \lambda_{0,j} + \nu_\eta$.

Here $D_\eta, C_\eta \rightarrow 1^+$ as $\eta \rightarrow 0$, and satisfy 2.12, 3.1 and 3.2 (see the remark at the end of section 3). Then we have

Lemma 4.2 : *As η goes to zero, $e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t)$ converges uniformly to 0 on $[t_1, +\infty[$ for any $t_1 > 0$.*

Proof: By Proposition 3.2 and Theorem 2.12, one has for $t_0 = t_0(\eta)$ and $t \geq t_0$

$$e^{\lambda_{0,j}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) - e^{\lambda_{0,j}t} \theta_{j,\nu}(t) \leq e^{\lambda_{0,j} C_\eta t} \theta_{j,\nu}(D_\eta^{-1} t) - e^{\lambda_{0,j}t} \theta_{j,\nu}(t) + \varepsilon(\eta).$$

Also by Proposition 3.1 and Theorem 2.11, one has for η small enough and $t_0 = t_0(\eta)$ and for $t \geq t_0$

$$e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) \leq e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j} D_\eta^{-1} t} \theta_{j,\nu}(C_\eta t) + \varepsilon(\eta).$$

So we deduce that for $t \geq t_0$

$$\left| e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) \right| \leq e^{\lambda_{0,j} C_\eta t} \theta_{j,\nu}(D_\eta^{-1} t) - e^{\lambda_{0,j} D_\eta^{-1} t} \theta_{j,\nu}(C_\eta t) + 2\varepsilon(\eta).$$

Let us estimate

$$\begin{aligned} I(t, \eta) &= e^{\lambda_{0,j} C_\eta t} \theta_{j,\nu}(D_\eta^{-1} t) - e^{\lambda_{0,j} D_\eta^{-1} t} \theta_{j,\nu}(C_\eta t) \\ &= t \int_{\lambda_{0,j} + \nu}^{\infty} e^{-(\lambda D_\eta^{-1} - \lambda_{0,j} C_\eta) t} N_j(\lambda) d\lambda - t \int_{\lambda_{0,j} + \nu}^{\infty} e^{-(\lambda C_\eta - \lambda_{0,j} D_\eta^{-1}) t} N_j(\lambda) d\lambda. \end{aligned}$$

Let

$$\alpha(\lambda, \eta) = \lambda D_\eta^{-1} - \lambda_{0,j} C_\eta$$

$$\text{and } \beta(\lambda, \eta) = \lambda C_\eta - \lambda_{0,j} D_\eta^{-1}.$$

First fix μ_0 so that $t_1 > 1/(\mu_0 - \lambda_{0,j})$. Then

$$I(t, \eta) = I_1(t, \eta) + I_2(t, \eta)$$

$$\text{where } I_1(t, \eta) = \int_{\lambda_{0,j} + \nu}^{\mu_0} t(e^{-\alpha(\lambda, \eta)t} - e^{-\beta(\lambda, \eta)t}) N_j(\lambda) d\lambda$$

$$\text{and } I_2(t, \eta) = \int_{\mu_0}^{\infty} t(e^{-\alpha(\lambda, \eta)t} - e^{-\beta(\lambda, \eta)t}) N_j(\lambda) d\lambda.$$

Let us first examine $I_2(t, \eta)$. For η near zero, the function $t \rightarrow t(e^{-\alpha(\lambda, \eta)t} - e^{-\beta(\lambda, \eta)t})$ is monotonic decreasing for $t > 1/\alpha$. Now, for η small enough, $1/\alpha < t_1$, and for $t \geq t_1$, the integrand in $I_2(t, \eta)$ is smaller than $t_1(e^{-\alpha(\lambda, \eta)t_1} - e^{-\beta(\lambda, \eta)t_1})$. Thus $I_2(t, \eta) < I_2(t_1, \eta)$, $t \geq t_1$ and we have to show the right hand side can be made small.

Now consider $r(\eta) = t_1(e^{-\alpha(\lambda, \eta)t_1} - e^{-\beta(\lambda, \eta)t_1})$ as a function of η . As $\eta \rightarrow 0$, $\alpha(\lambda, \eta)$ increases to $\lambda - \lambda_{0,j}$ while $\beta(\lambda, \eta)$ decreases to the same value. Thus $r(\eta)$ decreases to zero as η goes to zero. By the dominated convergence theorem, $I_2(t_1, \eta)$ goes to zero. We conclude that $I_2(t, \eta) \rightarrow 0$ *uniformly* in t , $t \geq t_1$ as $\eta \rightarrow 0$.

Next we estimate $I_1(t, \eta)$. Consider

$$f(t) = t e^{-\alpha(\lambda, \eta)t} (1 - e^{-(\beta(\lambda, \eta) - \alpha(\lambda, \eta))t})$$

for $t \in [0, \frac{2}{\alpha}]$. Then $\max_{t \in [0, +\infty]} f(t) = \max_{t \in [0, \frac{2}{\alpha}]} f(t)$ and

$$\begin{aligned} \max_{t \in [0, \frac{2}{\alpha}]} f(t) &\leq \max_{t \in [0, \frac{2}{\alpha}]} t e^{-\alpha(\lambda, \eta)t} \max_{t \in [0, \frac{2}{\alpha}]} (1 - e^{-(\beta - \alpha)t}) \\ &\leq \frac{1}{e \cdot \alpha} \frac{2}{\alpha} (\beta - \alpha). \end{aligned}$$

So

$$I_1(t, \eta) \leq \frac{2}{e} \int_{\lambda_{0,j} + \nu}^{\mu_0} \frac{(\beta(\lambda, \eta) - \alpha(\lambda, \eta))}{\alpha(\lambda, \eta)^2} N_j(\lambda) d\lambda.$$

Now

$$\beta(\lambda, \eta) - \alpha(\lambda, \eta) = (C_\eta - D_\eta^{-1})(\lambda + \lambda_{0,j}).$$

Moreover, since $\beta_j(M, g) > 0$, Lemma 1.2 tells us that

$$N_j(\lambda) \leq C(\lambda - \lambda_{0,j})^\beta,$$

for $\beta \in]0, 1[$ and $\lambda \leq \lambda_{0,j} + 1$. We first estimate

$$\frac{2}{e} \int_{\lambda_{0,j}+\nu}^{\lambda_{0,j}+1} \frac{(\beta(\lambda, \eta) - \alpha(\lambda, \eta))}{\alpha(\lambda, \eta)^2} N_j(\lambda) d\lambda \leq \frac{\bar{C}}{1-\beta} (2\lambda_{0,j} + 1) (C_\eta - D_\eta^{-1}) (\nu_\eta^{\beta-1} - 1).$$

Now assumption (3) implies that $\frac{(C_\eta - D_\eta^{-1})}{\nu_\eta}$ is bounded from above. Using this we see that

$$\frac{2}{e} \int_{\lambda_{0,j}+\nu}^{\lambda_{0,j}+1} \frac{(\beta(\lambda, \eta) - \alpha(\lambda, \eta))}{\alpha(\lambda, \eta)^2} N_j(\lambda) d\lambda \leq \frac{\bar{C}}{1-\beta} (2\lambda_{0,j} + 1) (C\nu_\eta^\beta - (C_\eta - D_\eta^{-1})).$$

This is a uniform estimate in t , $t \geq t_1$, and the right hand side goes to zero as $\eta \rightarrow 0$. We next estimate

$$\frac{2}{e} \int_{\lambda_{0,j}+1}^{\mu_0} \frac{(\beta(\lambda, \eta) - \alpha(\lambda, \eta))}{\alpha(\lambda, \eta)^2} N_j(\lambda) d\lambda \leq C(\lambda_{0,j} + \mu_0) (C_\eta - D_\eta^{-1}) N_j(\mu_0) (1 - (\mu_0 - \lambda_{0,j})^{-1}).$$

This is a uniform estimate in t , $t \geq t_1$, and the right hand side goes to zero as $\eta \rightarrow 0$. Combining these estimates, we conclude that $I_1(t, \eta) \rightarrow 0$ *uniformly* in t , $t \geq t_1$ as $\eta \rightarrow 0$. This completes the proof of the lemma.

Lemma 4.3 : *As η goes to zero, $e^{\lambda_{0,j}t} \theta_{j,\nu}(t)$ converges uniformly to $e^{\lambda_{0,j}t} \theta_j(t)$ on $[0, +\infty[$.*

Proof: We estimate

$$|e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j}t} \theta_j(t)| = t \int_{\lambda_{0,j}}^{\lambda_{0,j}+\nu} e^{-(\lambda - \lambda_{0,j})t} N_j(\lambda) d\lambda.$$

Since the function $t \rightarrow te^{-(\lambda - \lambda_{0,j})t}$ has its maximum at $t = \frac{1}{\lambda - \lambda_{0,j}}$, we see that

$$\begin{aligned} |e^{\lambda_{0,j}t} \theta_{j,\nu}(t) - e^{\lambda_{0,j}t} \theta_j(t)| &\leq e^{-1} \int_{\lambda_{0,j}}^{\lambda_{0,j}+\nu} \frac{N_j(\lambda)}{(\lambda - \lambda_{0,j})} d\lambda \\ &\leq C \frac{e^{-1}}{\beta} \nu_\eta^\beta \quad \text{where } \beta > 0, \text{ since } \beta_j(M, g) > 0 \text{ and by Lemma 1.2.} \end{aligned}$$

This proves the lemma.

Lemma 4.4 : *As η goes to zero, $e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) - e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}}t} \theta_j^{\tilde{K}, \tilde{W}}(t)$ converges uniformly to zero on $[0, +\infty[$.*

Proof: We estimate

$$|e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}}t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) - e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}}t} \theta_j^{\tilde{K}, \tilde{W}}(t)| = t \int_{\lambda_{0,j}^{\tilde{K}, \tilde{W}}}^{\lambda_{0,j}^{\tilde{K}, \tilde{W}}+\nu} e^{-(\lambda - \lambda_{0,j}^{\tilde{K}, \tilde{W}})t} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda.$$

By Proposition 2.10, $N_j^{\tilde{K}, \tilde{W}}(\lambda) \leq N_j(D_\eta \lambda)$, and because $\beta_j(M, g) > 0$,

$$N_j(\lambda) \leq C(\lambda - \lambda_{0,j})^\beta,$$

for $\beta > 0$ and $\lambda \leq \lambda_{0,j} + 1$. Therefore

$$\begin{aligned} |e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_{j,\nu}^{\tilde{K}, \tilde{W}}(t) - e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t)| &\leq C t e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \int_{\lambda_{0,j}^{\tilde{K}, \tilde{W}}}^{\lambda_{0,j} + \nu} e^{-\lambda t} (D_\eta \lambda - \lambda_{0,j})^\beta d\lambda \\ &\leq C t e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} (D_\eta(\lambda_{0,j} + \nu) - \lambda_{0,j})^\beta \int_{\lambda_{0,j}^{\tilde{K}, \tilde{W}}}^{\lambda_{0,j} + \nu} e^{-\lambda t} d\lambda \\ &\leq C (D_\eta(\lambda_{0,j} + \nu) - \lambda_{0,j})^\beta (1 - e^{-\nu t}). \end{aligned}$$

This proves the lemma.

This also completes the proof of Theorem 4.1.

Remarks. Suppose one could prove the following result,

Assume that $\beta_j(M, g) > 0$. Given small $\varepsilon > 0$, there are positive constants C_1 and C_2 which are independent of the triangulation such that

$$C_1 t^{-\bar{\beta}_j(M, g) - \varepsilon} \leq \theta_j(t) \leq C_2 t^{-\beta_j(M, g) + \varepsilon}.$$

Then, combined with Theorem 4.1, one could deduce a conjecture stated in the introduction, that is,

Assume that $\beta_j(M, g) > 0$. Then $\beta_j(K, g)$ converges to $\beta_j(M, g)$ as the mesh of the triangulation goes to zero. Here $\beta_j(K, g)$ denotes the combinatorial counterpart of $\beta_j(M, g)$.

We are sadly as yet unable to improve the estimates in this section to prove these conjectures.

5 Calculations

In this section, we calculate our von Neumann spectral invariants $\beta_j(X, g)$ on closed hyperbolic manifolds, showing that they differ in general from the Novikov-Shubin invariants. We also define Riemannian manifolds (M, g) with positive β -decay, that is, $\beta_j(M, g) > 0$ is positive for all j . We prove a result which gives some evidence to the conjecture stated in the introduction.

Let X be a closed hyperbolic manifold of dimension $d = 2n + 1$. This means that $X = \Gamma \backslash G / K$ is a rank one locally symmetric space with $G = SO_0(1, d)$, $K = SO(d)$ and $\Gamma \subset G$ is a torsion-free co-compact

discrete subgroup. We shall use some results of Fried [11] to help us compute our invariants for X . We first describe the Laplacian Δ_j on j -forms on G/K in group theoretic terms as in [11].

It turns out that if one normalises the Killing form c on the Lie algebra \mathcal{G} of G to $\frac{1}{2d-2}c$, then the induced G invariant metric on G/K has constant sectional curvature equal to -1 .

Also the Casimir operator on G induces a Casimir operator Ω_j on G/K acting on the space of L^2 sections of the homogeneous vector bundle of j -forms on G/K . Then the Laplacian $\Delta_j = -\frac{1}{2d-2}\Omega_j$ is a constant multiple of the Casimir operator.

Consider the Iwasawa decomposition $G = KAN$ where A has dimension one and hence G/K is a rank one symmetric space such that the rank of G is greater than the rank of K . Let M be the centraliser of A in K .

G has no discrete series representations and the principal series representations of G are parametrised by $\hat{M} \times \mathbb{R}$, which carries a smooth Plancherel density [13]. On each line $\sigma \times \mathbb{R}$, it is of the form $P_\sigma(\nu)d\nu$, where $P_\sigma(\nu)$ is an even polynomial of degree $d-1$.

The σ 's which are of interest to us are those which occur in the restriction of ξ_j to the subgroup M , where ξ_j denotes the usual representation of $K = SO(d)$ on $\Lambda^j \mathcal{C}^d$. Writing $\mathcal{C}^d = \mathcal{C}^{d-1} \oplus \mathcal{C}$, we observe that each $\omega \in \Lambda^j \mathcal{C}^d$ is of the form $\omega' + \omega'' \wedge dx_d$ where $\omega' \in \Lambda^j \mathcal{C}^{d-1}$ and $\omega'' \in \Lambda^{j-1} \mathcal{C}^{d-1}$. Hence ξ_j restricted to M is isomorphic to $\sigma_j \oplus \sigma_{j-1}$, where σ_j is the usual representation of $M = SO(d-1)$ on $\Lambda^j \mathcal{C}^{d-1}$. Each σ_j is unitary and irreducible except in the case when $j = n$, in which case it decomposes as a direct sum of two irreducible representations σ_j^+ and σ_j^- .

The following theorem can be deduced from [11], theorem 2, observing that the von Neumann trace of the heat kernel on j -forms on G/K is just the identity term in Fried's version of the Selberg Trace Formula for the trace of the heat kernel on j -forms on X .

Theorem 5.1 *For $j = 0, 1, \dots, n$ we have*

$$\theta_j(t) = \tau(\exp(-t\Delta_j)) = I_t(\sigma_j) + I_t(\sigma_{j-1})$$

where $I_t(\sigma_{-1}) \equiv 0$ and

$$I_t(\sigma_j) = a_j \int_{-\infty}^{\infty} \exp(-t(\nu^2 + c_j^2)) P_{\sigma_j}(\nu) d\nu$$

Here $a_j = \binom{d-1}{j} \text{vol}(X)$ and $c_j = n - j$.

By the isometry induced by the Hodge star operator, we see that

$$\theta_j(t) = \theta_{d-j}(t)$$

for $j = 0, 1, 2, \dots, n$ and hence we obtain expressions for $\theta_j(t)$ for $j = 0, 1, \dots, d$. Using these, and the explicit expression for the Plancherel measure [13], [19], we will be able to compute $\beta_j(X)$.

Theorem 5.2 . *Let X be a closed hyperbolic manifold of dimension $d = 2n + 1$. Then $\overline{\beta}_j(X) = \beta_j(X) = \frac{3}{2}$ for $j = 0, 1, \dots, n-1$ and $\overline{\beta}_n(X) = \beta_n(X) = \frac{1}{2}$.*

Proof. Using the following explicit expression for the Plancherel measure,

$$P_{\sigma_j}(\nu) = \left(\nu^2 + (n-j)^2\right)^{-1} \prod_{k=0}^n \left(\nu^2 + k^2\right)$$

we see that for $j \leq n-1$,

$$\begin{aligned} I_t(\sigma_j) &= C_1 \exp(-tc_j^2) \left\{ \int_{-\infty}^{\infty} \exp(-t\nu^2) \nu^2 d\nu + O(t^{-\frac{5}{2}}) \right\} \\ &= C'_1 \exp(-tc_j^2) \left\{ t^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}) \right\} \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Using the previous theorem, we see that $\overline{\beta}_j(X) = \beta_j(X) = \frac{3}{2}$ for $j = 0, 1, \dots, n-1$.

The case when $j = n$ remains to be studied. In this case

$$I_t(\sigma_n) = C_2 \int_{-\infty}^{\infty} \exp(-t\nu^2) d\nu + O(t^{-\frac{5}{2}}) = C'_2 t^{-\frac{1}{2}} + O(t^{-\frac{3}{2}}).$$

Using the previous theorem, we see that $\overline{\beta}_n(X) = \beta_n(X) = \frac{1}{2}$.

Definition 5.3 . A closed Riemannian manifold (M, g) is said to have positive β -decay if $\beta_j(M, g) > 0$ is positive for all j .

We recall that a closed Riemannian manifold (M, g) is said to be L^2 -acyclic if all the L^2 Betti numbers of its universal cover vanish, that is, $b_{(2)}^j(\widetilde{M}) = 0$ for all j . (cf. [16], [4].)

Proposition 5.4 . Let (M, g) be a closed L^2 acyclic Riemannian manifold with positive β -decay, and (N, h) be any closed L^2 acyclic Riemannian manifold. Then $(M \times N, g \times h)$ has positive β -decay.

Proof. Since $M \times N$ is given the product metric $g \times h$,

$$\tau_{M \times N} \left(e^{-t\Delta_k^{M \times N}} \right) = \sum_{i+j=k} \tau_M \left(e^{-t\Delta_i^M} \right) \tau_N \left(e^{-t\Delta_j^N} \right)$$

and

$$\lambda_{0,k}^{M \times N} = \text{minimum} \left\{ \lambda_{0,i}^M + \lambda_{0,j}^N : i+j=k \right\}.$$

Since M and N are L^2 -acyclic

$$\theta_k^{M \times N}(t) = \sum_{i+j=k} \theta_i^M(t) \theta_j^N(t).$$

Hence we see that

$$\beta_k(M \times N, g \times h) = \text{minimum} \left\{ \beta_i(M, g) + \beta_j(N, h) : i+j=k \right\}$$

The theorem follows.

Corollary 5.5 . Let (M, g) be a closed, odd dimensional, hyperbolic manifold with positive β -decay, and (N, h) be any closed L^2 acyclic Riemannian manifold. Then $(M \times N, g \times h)$ has positive β -decay.

6 Von Neumann Determinants and β -Torsion

In this section, we assume that (M, g) is a closed n -dimensional Riemannian manifold such that $\beta_j(M, g) > 0$. We will define the von Neumann determinant of the operator $\Delta_j - \lambda_{0,j}$ on \widetilde{M} following ideas of [16] and [14], where the von Neumann determinant of the operator Δ_j was studied. Hence we will also assume that $\lambda_{0,j} > 0$, without any loss of generality. We will compute these determinants for certain closed hyperbolic dimensional manifolds. We also define the analytic β -torsion in terms of the von Neumann determinants of the operators $\Delta_j - \lambda_{0,j}$, by analogy to Ray-Singer torsion [18]. We also define its combinatorial counterpart, which we call combinatorial β -torsion. In the next section, we prove a result which gives evidence that the combinatorial β -torsion converges to the analytic β -torsion, as the mesh of the triangulation goes to zero.

We begin by defining the partial L^2 zeta functions of the operator $\Delta_j - \lambda_{0,j}$ as follows.

Definition 6.1 . *The partial L^2 zeta functions of the operator $\Delta_j - \lambda_{0,j}$ are*

$$\begin{aligned}\zeta_j^{(1)}(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} e^{\lambda_{0,j}t} \theta_j(t) dt \\ \text{and } \zeta_j^{(\infty)}(s) &= \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} e^{\lambda_{0,j}t} \theta_j(t) dt.\end{aligned}$$

Here s belongs to a subset of complex numbers which will be specified shortly.

We begin by proving the following lemma.

Lemma 6.2 . *$\zeta_j^{(1)}(s)$ is a holomorphic function in the half-plane $\Re(s) > n/2$ and has a meromorphic continuation to \mathbb{C} with no pole at $s = 0$.*

Proof. Using [20], we have an asymptotic expansion as $t \rightarrow 0^+$ of $\tau(e^{-t\Delta_j})$,

$$\tau(e^{-t\Delta_j}) \sim t^{-n/2} \sum_{j=0}^{\infty} t^j c_j \quad \text{as } t \rightarrow 0^+.$$

Hence $e^{\lambda_{0,j}t} \tau(e^{-t\Delta_j})$ has the following small time asymptotic expansion

$$e^{\lambda_{0,j}t} \tau(e^{-t\Delta_j}) \sim t^{-n/2} \sum_{i=0}^{\infty} \sum_{l+k=i} c_l \frac{\lambda_{0,j}^k}{k!} t^i \quad \text{as } t \rightarrow 0^+.$$

In particular, $e^{\lambda_{0,j}t} \tau(e^{-t\Delta_j}) \leq C t^{-n/2}$ for $0 \leq t \leq 1$. We deduce that $\zeta_j^{(1)}(s)$ is well defined on the half-plane $\Re(s) > n/2$. Clearly

$$\frac{\partial}{\partial \bar{s}} \zeta_j^{(1)}(s) = 0$$

in this region, that is, $\zeta_j^{(1)}(s)$ is holomorphic in this half-plane.

The meromorphic continuation of $\zeta_j^{(1)}(s)$ to the half-plane $\Re(s) > n/2 - N$ is obtained by considering the first N terms of the small time asymptotic expansion of $e^{\lambda_{0,j}t}\tau(e^{-t\Delta_j})$,

$$\begin{aligned}\zeta_j^{(1)}(s) &= -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) \frac{b_{(2)}^j(\widetilde{M})}{s\Gamma(s)} + \frac{1}{\Gamma(s)} \left[\int_0^1 t^{s-1-n/2} \left(\sum_{i=0}^N \sum_{l+k=i} c_l \frac{\lambda_{0,j}^k}{k!} t^i \right) dt + R_N(s) \right] \\ &= -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) \frac{b_{(2)}^j(\widetilde{M})}{s\Gamma(s)} + \frac{1}{\Gamma(s)} \left[\sum_{i=0}^N \frac{\sum_{l+k=i} c_l \frac{\lambda_{0,j}^k}{k!}}{(s+i-n/2)} + R_N(s) \right]\end{aligned}$$

where $R_N(s)$ is holomorphic in the half plane $\Re(s) > n/2 - N$.

By observation, we see that the meromorphic continuation of $\zeta_j^{(1)}(s)$, also denoted by $\zeta_j^{(1)}(s)$, has no pole at $s = 0$.

Corollary 6.3 . $\zeta_j^{(1)}(0) = \begin{cases} -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) b_{(2)}^j(\widetilde{M}) + \sum_{l+k=\frac{n}{2}} c_l \frac{\lambda_{0,j}^k}{k!} & \text{if } n \text{ is even} \\ -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) b_{(2)}^j(\widetilde{M}) & \text{if } n \text{ is odd.} \end{cases}$

Lemma 6.4 . $\zeta_j^{(\infty)}(s)$ is holomorphic in the half-plane $\Re(s) < \beta_j(M, g)$.

Proof. Using the estimate $e^{\lambda_{0,j}t}\theta_j(t) \leq Ct^{-\beta_j(M,g)+\varepsilon}$ we see that $\zeta_j^{(\infty)}(s)$ is well defined on the half-plane $\Re(s) < \beta_j(M, g)$, since

$$\left| \zeta_j^{(\infty)}(s) \right| \leq \frac{C}{|\Gamma(s)(\Re(s) - \beta_j(M, g))|}$$

whenever $\Re(s) < \beta_j(M, g)$. Clearly $\frac{\partial}{\partial \bar{s}} \zeta_j^{(\infty)}(s) = 0$ on this half-plane.

The following is an immediate consequence of the proof of Lemma 6.4.

Corollary 6.5 . $\zeta_j^{(\infty)}(0) = 0$.

Definition 6.6 . Define the L^2 zeta function of the operator $\Delta_j - \lambda_{0,j}$ as follows.

$$\zeta_j(s) = \zeta_j^{(1)}(s) + \zeta_j^{(\infty)}(s).$$

The following theorem summarizes the prior lemmas.

Theorem 6.7 . $\zeta_j(s)$ is holomorphic near $s = 0$. Its value at zero is

$$\zeta_j(0) = \begin{cases} -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) b_{(2)}^j(\widetilde{M}) + \sum_{l+k=\frac{n}{2}} c_l \frac{\lambda_{0,j}^k}{k!} & \text{if } n \text{ is even} \\ -\left(\frac{e^{\lambda_{0,j}} - 1}{\lambda_{0,j}}\right) b_{(2)}^j(\widetilde{M}) & \text{if } n \text{ is odd.} \end{cases}$$

Definition 6.8 . *The von Neumann determinant of the operator $\Delta_j - \lambda_{0,j}$ on \widetilde{M} is by definition*

$$|Det_\tau|(\Delta_j - \lambda_{0,j}) = \exp(-\zeta'_j(0)).$$

We now compute this determinant on hyperbolic manifolds. In principle, it is possible to use a computer program and compute all the determinants. As an illustration, we compute the von Neumann determinant of $\Delta_1 - \lambda_{0,1} = \Delta_1 - 1$ on five dimensional hyperbolic space. We use the notation of section 5.

Since there are no L^2 harmonic differential forms on five dimensional hyperbolic space, we see that $\zeta_j(0) = 0$ by Theorem 6.7. By Theorem 5.1,

$$\theta_1(t) = \tau(\exp(-t\Delta_1)) = I_t(\sigma_1) + I_t(\sigma_0)$$

where

$$I_t(\sigma_j) = a_j \int_{-\infty}^{\infty} \exp(-t(\nu^2 + c_j^2)) P_{\sigma_j}(\nu) d\nu$$

Here $j = 0, 1$, $a_j = \binom{4}{j} \text{vol}(X)$ and $\lambda_{0,j} = c_j^2 = (2-j)^2$.

An easy calculation yields

Lemma 6.9 . *The von Neumann determinant of $\Delta_1 - \lambda_{0,1}$ is given by*

$$\log |Det_\tau|(\Delta_1 - 1) = 8.7062 \text{ vol}(X)$$

Next we define the analytic β -torsion to be

Definition 6.10 . *Suppose that M is a manifold with positive β -decay. Then the analytic β -torsion is defined to be the product*

$$T(\widetilde{M}, g) = \prod_{j=0}^n |Det_\tau|(\Delta_j - \lambda_{0,j})^{j(-1)^j}$$

One can make an analogous definition in the combinatorial setting. The results of the next section suggest that the combinatorial torsion may converge to its analytic counterpart. We plan to discuss this further elsewhere.

7 Convergence of Theta and Zeta Functions

Using the results of section 3, we prove in this section that the L^2 combinatorial theta function, as a function of the time variable converges uniformly on compact subsets to the L^2 analytic theta function as the mesh of the triangulation goes to zero.

Then we are able to prove the analogue of this result for L^2 -zeta functions. However, to define these zeta functions, one needs to impose the condition of positive decay on the manifold. Our method of circumventing the slow decay of the L^2 theta functions is to split the L^2 zeta functions into two partial zeta functions. We introduce an analogous splitting of the combinatorial L^2 zeta function and prove that the resulting partial zeta functions converge as the mesh of the triangulation goes to zero to the corresponding L^2 analytic partial zeta functions, uniformly on compact subsets of their respective domains. These results are analogs for our situation of the Dodziuk-Patodi theorems [8].

If our manifold has positive β -decay, then using Theorem 4.1, we prove that the zeta functions of the operator $\Delta_j^{\tilde{K}, \tilde{W}} - \lambda_{0,j}^{\tilde{K}, \tilde{W}}$ converge to the zeta functions of the operator $\Delta_j - \lambda_{0,j}$ as the mesh of the triangulation goes to zero. Recall that we have also come across these zeta functions in the previous section.

We begin with

Theorem 7.1 : *As the mesh goes to zero, $\theta_j^{\tilde{K}, \tilde{W}}(t)$ converges to $\theta_j(t)$ uniformly on the set $[t_1, \infty)$ for any $t_1 > 0$.*

Proof: By Propositions 3.1 and 3.2, we see that

$$\left| \theta_j^{\tilde{K}, \tilde{W}}(t) - \theta_j(t) \right| \leq |\theta_j(D_\eta^{-1}t) - \theta_j(t)| + |\theta_j(C_\eta t) - \theta_j(t)| + \varepsilon(\eta).$$

So it suffices to prove that $\theta_j(B_\eta t)$ converges, uniformly in $t \in [t_1, \infty)$, to $\theta_j(t)$ as $\eta \rightarrow 0$. Here $B_\eta \rightarrow 1^+$ as $\eta \rightarrow 0$. By the mean value theorem, we see that for $t \geq t_1$,

$$|\theta_j(B_\eta t) - \theta_j(t)| \leq |B_\eta - 1|M$$

where $M = \sup_{t \geq t_1} |t \tau(\Delta_j e^{-t\Delta_j})|$. It suffices to show that M is finite.

Now,

$$\begin{aligned} t \tau(\Delta_j e^{-t\Delta_j}) &= t \int_0^\infty \lambda e^{-t\lambda} dN_j(\lambda) \\ &= -t \int_0^\infty e^{-t\lambda} N_j(\lambda) d\lambda + t^2 \int_0^\infty \lambda e^{-t\lambda} N_j(\lambda) d\lambda \\ &= -\tau(e^{-t\Delta_j}) + \int_0^\infty \lambda e^{-\lambda} N_j\left(\frac{\lambda}{t}\right) d\lambda. \end{aligned}$$

So for $t \geq t_1$,

$$|t \tau(\Delta_j e^{-t\Delta_j})| \leq b_{(2)}^j(\tilde{M}) + \theta_j(t_1) + \int_0^\infty \lambda e^{-\lambda} N_j\left(\frac{\lambda}{t_1}\right) d\lambda.$$

That is, $M < \infty$.

Let M denote a closed manifold. Recall the definition of the Novikov-Shubin invariants,

$$\alpha_j(M) = \sup\{\beta \in \mathbb{R} : \theta_j(t) \text{ is } O(t^{-\beta}) \text{ as } t \rightarrow \infty\} \in [0, \infty].$$

It has been shown by Novikov and Shubin [10] that the numbers $\alpha_j(M)$ are independent of the choice of metric on M , and in fact Gromov and Shubin [12] show that they depend only on the homotopy type of M . We recall from [16] the following definition.

Definition 7.2 : *A manifold M is said to have positive decay if all of its Novikov-Shubin invariants $\alpha_j(M)$ are positive, for $j \geq 0$.*

Let K be a smooth triangulation of M . Recall the definition of the combinatorial analogue of the Novikov-Shubin invariants,

$$\alpha_j(K) = \sup\{\beta \in \mathbb{R} : \theta_j^{\tilde{K}, \tilde{W}}(t) \text{ is } O(t^{-\beta}) \text{ as } t \rightarrow \infty\} \in [0, \infty].$$

It has been shown by A.V. Efremov [9] that the numbers $\alpha_j(K)$ are independent of the choice of triangulation of M , and hence they depend only on the topology of M . In fact, Efremov proves that

$$\alpha_j(K) = \alpha_j(M).$$

Note that one can extract the proof of his result from sections 2, 3 and 4.

See [16, 17], [14] for numerous examples of manifolds with positive decay.

In the following discussion, we will only consider manifolds with positive decay.

Now we define the zeta function of the Laplacian Δ_j following [16], and recall some of its properties. Introduce two partial zeta functions:

$$\zeta_j(s, 1) = \frac{1}{\Gamma(s)} \int_0^1 \theta_j(t) t^{s-1} dt, \quad (\Re(s) > \frac{1}{2} \dim M),$$

and under the assumption that M has positive decay,

$$\zeta_j(s, \infty) = \frac{1}{\Gamma(s)} \int_1^\infty \theta_j(t) t^{s-1} dt \quad (\Re(s) < \alpha_j(M)).$$

Then $\zeta_j(s, \infty)$ is analytic for $\Re(s) < \alpha_j(M)$ whereas $\zeta_j(s, 1)$ is analytic for $\Re(s) > \frac{1}{2} \dim M$ and has an analytic continuation to a neighbourhood of zero. Thus with this understanding we can define the zeta function to be the sum

$$\zeta_j(s) = \zeta_j(s, \infty) + \zeta_j(s, 1).$$

It is analytic on a neighbourhood of zero.

We have seen that the assumption of positive decay implies decay for $\theta_j^{\tilde{K}, \tilde{W}}(t)$ (with the same $\alpha_j(M)$) and hence we may define zeta functions $\zeta_j^{\tilde{K}, \tilde{W}}(s)$ similarly in terms of $\theta_j^{\tilde{K}, \tilde{W}}(t)$ (note however that in the combinatorial case the theta functions are well defined at $t = 0$ and so the analytic continuation we employed above is not necessary).

Theorem 7.3 . Let M be a closed manifold with positive decay. Then
1. the combinatorial partial zeta function

$$\zeta_j^{\tilde{K}, \tilde{W}}(s, 1) \equiv \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \theta_j^{\tilde{K}, \tilde{W}}(t) dt$$

converges for $\Re(s) > \frac{1}{2} \dim M = \frac{n}{2}$ uniformly on compact subsets to

$$\zeta_j(s, 1) \equiv \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \theta_j(t) dt$$

2. while the other combinatorial partial zeta function:

$$\zeta_j^{\tilde{K}, \tilde{W}}(s, \infty) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \theta_j^{\tilde{K}, \tilde{W}}(t) dt$$

converges for $\Re(s) < \alpha(M) = \min\{\alpha_j(M)\}$ uniformly on compact subsets to

$$\zeta_j(s, \infty) \equiv \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \theta_j(t) dt.$$

Proof. Part 1. Let F be a compact subset of the half-plane $\Re s > \frac{n}{2}$, that is $\Re s \geq \frac{n}{2} + \delta$ for all $s \in F$. We need to prove that given $\varepsilon > 0$

$$\int_0^1 dt \left| \int_0^\infty t^s e^{-t\lambda} \left(N_j(\lambda) - N_j^{\tilde{K}, \tilde{W}}(\lambda) \right) d\lambda \right| < \varepsilon.$$

for all η small enough and for any $s \in F$.

Since $\theta_j(t) \leq Ct^{-\frac{n}{2}}$ for $0 < t \leq 1$, one sees that

$$|\Gamma(s) \zeta_j(s, 1)| \leq \int_0^1 dt \int_0^\infty t^{\Re(s)} e^{-t\lambda} N_j(\lambda) d\lambda < \infty$$

is finite and bounded for any $s \in F$. Therefore, given $\varepsilon > 0$, there is an $m > 0$ such that

$$\int_0^1 dt \int_m^\infty t^{\Re(s)} e^{-t\lambda} N_j(\lambda) d\lambda < \varepsilon.$$

By the uniform convergence theorem and using Propositions 2.2 and 2.6, one has for η small enough,

$$\int_0^1 dt \left| \int_0^m t^s e^{-t\lambda} \left(N_j(\lambda) - N_j^{\tilde{K}, \tilde{W}}(\lambda) \right) d\lambda \right| < \varepsilon$$

uniformly for $s \in F$. In the notation of Proposition 2.2, define the interval

$$S_m = \left(m, \left\{ \frac{1}{C_1 \sqrt{h}} - 1 \right\}^2 \right)$$

where $h = (\eta |\log \eta|)^2$. Then by Proposition 2.2, for all $\lambda \in S_m$ one has the inequality

$$N_j^{\tilde{K}, \tilde{W}}(\lambda) \leq N_j(D_\eta \lambda).$$

On the other hand, if $\lambda > m$ and $\lambda \notin S_m$, then there is a $\beta > 0$ such that

$$\lambda \geq \beta(\eta |\log \eta|)^{-2}$$

for all η small enough. By the fullness assumption on K , one has

$$N_j^{\tilde{K}, \tilde{W}}(\lambda) \leq \gamma \eta^{-n}$$

for some $\gamma > 0$.

One estimates for $s \in F$,

$$\begin{aligned} \int_0^1 dt \left| \int_m^\infty t^s e^{-\lambda t} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda \right| &\leq \int_0^1 dt \left| \int_{\lambda \in S_m} t^{\Re(s)} e^{-\lambda t} N_j(D_\eta \lambda) d\lambda \right| \\ &+ \int_0^1 dt \left| \int_{\frac{\beta}{h}}^\infty t^{\Re(s)} e^{-\lambda t} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda \right| \\ &\leq D_\eta^{\Re(s)} \varepsilon + \gamma \beta^{-\Re(s)} \eta^{2\delta} |\log \eta|^2 e^{\Re(s)}. \end{aligned}$$

Thus there is a constant $C > 0$ such that for $s \in F$, one has

$$\int_0^1 dt \left| \int_m^\infty t^s e^{-\lambda t} N_j^{\tilde{K}, \tilde{W}}(\lambda) d\lambda \right| \leq C\varepsilon$$

for all η small enough.

This proves Part 1.

Part 2. Let F be a compact subset of the half-plane $\Re(s) < \alpha_j(M)$, that is $\Re(s) \leq \alpha_j(M) + \delta$ for all $s \in F$. We need to prove that given $\varepsilon > 0$

$$\int_1^\infty t^{\Re(s)-1} \left| \theta_j^{\tilde{K}, \tilde{W}}(t) - \theta_j(t) \right| dt < \varepsilon$$

for all η small enough. For $s \in F$, we see as in the first line of the proof of Theorem 7.1, that the sequence of functions

$$t^{\Re(s)-1} \left| \theta_j^{\tilde{K}, \tilde{W}}(t) - \theta_j(t) \right|$$

is dominated, for $t \in [1, \infty)$, by the function

$$t^{\Re(s)-1} \left| \theta_j\left(\frac{t}{2}\right) - \theta_j(t) \right| + t^{\Re(s)-1} |\theta_j(2t) - \theta_j(t)| + t^{\Re(s)-1} \varepsilon(t). \quad (*)$$

The first two terms above are clearly integrable on the interval $[1, \infty)$. We now examine the last term. From section 3, we see that

$$t^{\Re(s)-1} \varepsilon(t) \leq 2t^{\Re(s)} \int_\mu^\infty e^{-2t\lambda} N_j(\lambda) d\lambda + \frac{1}{2} t^{\Re(s)} \int_\mu^\infty e^{-\frac{t\lambda}{2}} N_j(\lambda) d\lambda.$$

We observe that for $a > 0$, one has

$$\int_1^\infty t^{\Re(s)} \int_\mu^\infty e^{-at\lambda} N_j(\lambda) d\lambda \leq \int_1^\infty t^{\Re(s)-1} \theta_j(at) dt.$$

Therefore for $s \in F$, one has

$$\int_1^\infty t^{\Re(s)-1} \varepsilon(t) dt \leq 2 \int_1^\infty t^{\Re(s)-1} \theta_j(2t) dt + \frac{1}{2} \int_1^\infty t^{\Re(s)-1} \theta_j\left(\frac{t}{2}\right) dt < \infty$$

and the third term in $(*)$ is integrable on the interval $[1, \infty)$. By Theorem 7.1, we see that

$$\lim_{\eta \rightarrow 0} t^{\Re(s)-1} \left| \theta_j^{\tilde{K}, \tilde{W}}(t) - \theta_j(t) \right| = 0$$

pointwise, for any $t \in [1, \infty)$. By the Dominated Convergence Theorem, we conclude that for $s \in F$

$$\int_1^\infty t^{\Re(s)-1} \left| \theta_j^{\tilde{K}, \tilde{W}}(t) - \theta_j(t) \right| dt < \varepsilon$$

for all η small enough, proving part 2.

This is the analogue for L^2 -zeta functions of the results of [8] for the ordinary zeta function. Notice that in fact the proofs make no essential use of the representation of the fundamental group (they use only the fact that the commutant is a semi-finite von Neumann algebra).

For the next theorem, we assume that the following conjecture holds,

Assume that $\beta_j(M, g) > 0$. Then $\beta_j(K, g)$ converges to $\beta_j(M, g)$ as the mesh of the triangulation goes to zero. Here $\beta_j(K, g)$ denotes the combinatorial counterpart of $\beta_j(M, g)$.

This conjecture was stated in the introduction and also discussed in section 4. Then in the notation of section 6, Theorem 4.1 combined with dominated convergence implies:

Theorem 7.4 . *Let (M, g) be a closed Riemannian manifold with positive β -decay. Then the combinatorial partial zeta function of the operator $\Delta_j^{\tilde{K}, \tilde{W}} - \lambda_{0,j}^{\tilde{K}, \tilde{W}}$*

$$\zeta_j^{\tilde{K}, \tilde{W}(1)}(s) \equiv \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t) dt$$

converges for $\Re(s) > \frac{1}{2} \dim M$ uniformly on compact subsets to

$$\zeta_j^{(1)}(s) \equiv \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} e^{\lambda_{0,j} t} \theta_j(t) dt$$

while the other combinatorial partial zeta function of the operator $\Delta_j^{\tilde{K}, \tilde{W}} - \lambda_{0,j}^{\tilde{K}, \tilde{W}}$:

$$\zeta_j^{\tilde{K}, \tilde{W}(\infty)}(s) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} e^{\lambda_{0,j}^{\tilde{K}, \tilde{W}} t} \theta_j^{\tilde{K}, \tilde{W}}(t) dt$$

converges for $\Re(s) < \beta = \min\{\beta_j(M, g)\}$ uniformly on compact subsets to

$$\zeta_j^{(\infty)}(s) \equiv \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} e^{\lambda_{0,j}t} \theta_j(t) dt.$$

Proof. The proof is similar to that given in the previous theorem, but now using the results in sections 2 and 4 instead. The only new point to observe is that the assumption that the conjecture holds is used to show that when the mesh of the triangulation is small enough, one sees that $\beta_j(K, g)$ is positive and so the combinatorial partial zeta function $\zeta_j^{\tilde{K}, \tilde{W}(\infty)}(s)$ is defined. The detailed proof will be omitted.

8 Appendix

Here we present a proof due to Terry Lyons that $\bar{\beta}_0(M, g) \geq 1$ and $\beta_0(M, g) \geq 1$. He has kindly permitted us to do so.

Proposition. $\bar{\beta}_0(M, g) \geq 1$ and $\beta_0(M, g) \geq 1$.

Proof. Let $\lambda_{0,0}$ be the spectral gap. Then there is a $v > 0$, $v \in L^\infty(\tilde{M})$ such that

$$\Delta_0 v = \lambda_{0,0} v.$$

That is, v is a ground state for Δ_0 which is unique up to a multiplicative constant. Then either there is a $\gamma \in \pi_1(M)$ such that $\gamma^* v$ is not proportional to v , or for every $\gamma \in \pi_1(M)$,

$$\gamma^* v = \varphi(\gamma) v$$

for some morphism $\varphi : \pi_1(M) \rightarrow \mathbb{R}_+$.

Case 1. Suppose that there is a $\gamma \in \pi_1(M)$ such that $\gamma^* v$ is not proportional to v . Then

$$u = \frac{\gamma^* v}{v} > 0$$

is a positive, non-constant harmonic function for the Markovian semi-group with integral kernel

$$\tilde{p}_t(x, y) = e^{\lambda_{0,0}t} p_t(x, y) \frac{v(y)}{v(x)}.$$

Since 1 and u are non-proportional positive harmonic functions for this semigroup, it is a classical result then that \tilde{p}_t is transient cf.[1] page 44, that is,

$$\int_1^\infty e^{\lambda_{0,0}t} p_t(x, x) dt < \infty.$$

Case 2. Suppose that for every $\gamma \in \pi_1(M)$,

$$\gamma^* v = \varphi(\gamma) v$$

for some morphism $\varphi : \pi_1(M) \rightarrow \mathbb{R}_+$. Then the Markovian semigroup

$$\tilde{p}_t(x, y) = e^{\lambda_{0,0}t} p_t(x, y) \frac{v(y)}{v(x)}$$

is $\pi_1(M)$ invariant, by a simple calculation. Now by [15] theorem 3, \tilde{p}_t admits a non-constant, positive harmonic function u . So we see again that $\tilde{p}_t(x, y)$ is transient.

We conclude that $\tilde{p}_t(x, y)$ is always transient. It follows immediately that $\bar{\beta}_0(M, g) \geq 1$. To prove that $\beta_0(M, g) \geq 1$, we further argue as follows.

Let $K = \int_1^\infty e^{\lambda_{0,0}t} p_t(x, x) dt < \infty$. Recall that the function $t \rightarrow e^{\lambda_{0,0}t} p_t(x, x)$ is non-increasing. Therefore

$$\frac{t}{2} e^{\lambda_{0,0}t} p_t(x, x) \leq \int_{\frac{t}{2}}^t e^{\lambda_{0,0}s} p_s(x, x) ds \leq K$$

and $\beta_0(M, g) \geq 1$.

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